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KLEIN'S EVANSTON LECTURES.

By PROF. GEORGE BRUCE HALSTED, Austin, Texas.

Sylvester told me that he and Kronecker in attempting a definition of mathematics could only get as far as agreeing that it was essentially a kind of poesy. Of this poetry Professor Ziwet has caught for us a book of sonnets by our master geometer Felix Klein. For transcribing these harmonies and thus making them permanently accessible to all the world, American mathematicians in particular should be profoundly grateful to Professor Ziwet.

The charm of these lectures is so manifold that no one can afford to deprive himself of the pleasure of reading them. Throughout we see the very man who shows that all our space-intuitions must be held subject to revision everywhere emphasizing the tremendous power of these very space-intuitions as instrument in all mathematical research.

It is this point of view that divides original mathematicians into three classes—logicians, formalists, and intuitionists. With truest tact Klein chooses as companion for our Cayley and Sylvester that terrific analyst Gordan, who first showed the non-existence of the obstacle which had stopped both of them, the supposed infinitude of invariant forms.

As geometer supreme of the moderns he gives not Steiner but *von Staudt*, my own ideal, still the man of the future, whose pure system has never yet been given in English, but must now be, since no other will serve as foundation for projective metrics and projective non-Euclidean geometry.

And here another name should suggest itself—Riemann; for though so powerfully equipped as an analyst that in the matter of primes he succeeded where all others had failed, yet was he of essence geometer. See page 6 for "one of the best illustrations of the utility of adopting Riemann's principles." In point of fact, this wonderfully gifted man can never be overestimated. Though modest, sweet-natured, and painfully shy, yet fortunately he had a gentle obstinacy which saved him from adopting, in regard to his own work, suggestions kindly given by less gifted men. So what his writings lacked in immediate acceptance and recognition was more than made up in their fundamental, wide-reaching, continuous influence on subsequent mathematical thought. Justly his fame, long great, still grows.

In the two lectures on Sophus Lie we meet a clear presentation of the application of geometry to analysis, where the power is increased by adopting Pluecker's idea of a generalized space-element.

In the preface to my Pure Projective Geometry I call attention to Newton's ability and achievement in that line, saying: "Newton showed the extraordinary correlating power of projection, for example in his *Enumeratis* *linearum tertii ordinis*, where he gives sixty-four of the different species as projections of five." Starting with nearly this text, Lecture IV outlines what has since been done toward giving actual mental images of algebraic curves and surfaces.

Lecture V is a beautiful illustration of how what is really elementary geometry can yet advance research in the theory of functions.

Lecture VI discusses the distinction between the *naïve* and the *refined* intuition. "It is the latter that we find in Euclid; he carefully develops his system on the basis of well-formulated axioms, is fully conscious of the necessity of exact proofs, clearly distinguishes between the commensurable and incommensurable, and so forth." "We are living in a *critical* period similar to that of Euclid." Professor Klein then speaks of "that artistic finish that we admire in Euclid's 'Elements,'" and mentions Allman's important historical work. I heartily concur in this estimate of Euclid, and desire to contrast it with the error of Charles S. Peirce, in the *Nation*, where he speaks of "Euclid's proof (Elements, Bk. I., props. 16 and 17)" as "really quite fallacious, because it uses no premises not as true in the case of spherics." Our bright American seems to have forgotten Euclid's Postulate 6 (Axiom 12 in Gregory, Axiom 9 in Heiberg), "Two straight lines cannot enclose a space;" that is, two straights having crossed never recur.

Professor Klein agrees with Clifford, "that *the naïve intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact.*" Yet these two men are alike in a marvellous, astoundingly powerful space-intuition. They were born to be geometers. Professor Klein says, "It seems to me therefore, that Kirchhoff makes a mistake when he says in his *Spectral-Analyse* that absorption takes place only where there is *exact* coincidence between the wave-lengths. I side with Stokes, who says that absorption takes place *in the vicinity* of such coincidence." This reminds us how easily Clifford swept away Maxwell's argument for special creation from coincidence in size of molecules.

Lecture VII on the transcendency of the numbers e and π, by its very simplicity brings home to us more sharply the lack in English of any adequate treatment of the continuity of the number system. Professor Fine has written a book on the number system without even attempting this its fundamental problem. He borrows his continuity bodily from space in the following sentence: "*The entire system of real numbers*, however, inasmuch as it contains an individual number to correspond to every individual point in the continuous series of points forming a right line, *is continuous*."

Lecture VIII, on ideal numbers, shows that geometry will simplify even the proud and exclusive theory of numbers.

In Lecture IX, on the solution of equations, the quintic groups the paladins of algebra. Beginning after Abel, Rowan Hamilton, Sylvester, yet in 8 pages we have the names of Galois, Hermite, Kronecker, Brioschi, Gordan, Camille Jordan, Burkhardt, and Klein ; and the last is the one geometer who has reduced the solution of the quintic to the simplest form, and that by connecting it with the icosahedron.

After Lecture X, on hyperelliptic and Abelian functions, the course closes with Lecture XI on the most recent researches in non-Euclidean geometry. Three points of view are distinguished ; that of elementary geometry, of which Lobatschewsky and Bolyai are representatives ; that of projective geometry, where it is essential to adopt the system of von Staudt ; and that of Riemann and Helmholtz, starting with the element of a geodesic.

Attention is then called to the fact that a curved tridimensional space does not need a higher space in which to be curved. The curvature is an intrinsic characteristic quite independent of any higher space. Similarly we are cautioned against concluding from the familiar and highly important example of surface spherics, that in elliptic space two geodesics issuing from any point meet again in an antipodal point. " The projective theory of non-Euclidean space shows immediately that the existence of an antipodal point, though compatible with the nature of an elliptic space, is not necessary, but that two geodesic lines in such a space may intersect in one point if at all."

An exposition is now given of how Sophus Lie has confirmed the results of Helmholtz, and then is stated the intensely interesting outcome reached by Professor Klein in accordance with Clifford's general idea presented at the Bradford meeting of the British Association. The whole theory has been since verified by Killing.

Five pages have been added, in the form of a twelfth lecture, of advice to American students contemplating study in Germany ; and by way of an Appendix to the whole, we have a translation of Professor Klein's charming sketch of the development of mathematics at the German universities. What is most characteristic in the present *Lehr-Freiheit* is there traced to Jacobi. " The new feature is that Jacobi lectured exclusively on those problems on which he was working himself, and made it his sole object to introduce his students into his own circle of ideas. With this end in view he founded, for instance, the first mathematical seminary."

And now, in taking leave of this inspiring book, I desire to express my feeling of personal obligation to Professor Ziwet for his part in its production, and to heartily recommend it, not only to every professed mathematician, but to all lovers of high thinking.

UNIVERSITY OF TEXAS, *February*, 1894.